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ON A CRITERION OF PRINGSHEIM'S FOR EXPANSIBILITY IN TAYLOR'S SERIES

BY M. B. PORTER

IN his paper on Taylor's series* Pringsheim has established, among others, the following criterion for the expansibility of a function in Taylor's series:

If we denote by C_n the Cauchy Remainder:

$$C_n = h^n f^{(n)}(x + \theta h) \frac{(1 - \theta)^{n-1}}{(n-1)!},$$

then the necessary and sufficient condition that

$$J(x + h) = \sum_0^{\infty} f^{(n)}(x) \frac{h^n}{n!} \quad (0 \leq h < R)$$

is that

$$(1) \quad \lim_{n=\infty} C_n = 0$$

uniformly for all values of θ and h such that $0 \leq \theta \leq 1$, $0 \leq h \leq R_1 < R$; or, as we may say, over the "rectangle" $[0, R_1; 0, 1]$.

The sufficiency of Pringsheim's criterion is obvious at once; the proof of its necessity is a more delicate matter. The following considerations furnish a simple and elementary proof of Pringsheim's criterion which is applicable without modification to the case when f is a complex function of a real variable.

* Zum Taylor'schen Lehrsatz, *Math. Ann.*, vol. 44 (1894), p. 68.

If we integrate the right-hand side of the identity

$$f(x+h) - f(x) = \int_0^h f'(x+h-t) dt$$

$n-1$ times by parts we get

$$f(x+h) = \sum_{m=0}^{n-1} f^{(m)}(x) \frac{h^m}{m!} + \int_0^h f^{(n)}(x+h-t) \frac{t^{n-1}}{(n-1)!} dt,$$

where the integral on the right is Lagrange's integral for the remainder. In this integral set

$$h-t = \theta h$$

and it becomes

$$R_n = h^n \int_0^1 f^{(n)}(x+\theta h) \frac{(1-\theta)^{n-1}}{(n-1)!} d\theta.$$

Thus, the necessary and sufficient condition that

$$\sum_0^{\infty} f^{(n)}(x) \frac{h^n}{n!}$$

converge to the limit $f(x+h)$ is that

$$\lim_{n \rightarrow \infty} R_n = 0.$$

Since a power series always converges uniformly over any interval lying inside its interval of convergence we have that

$$(2) \quad \lim_{n \rightarrow \infty} R_n = 0$$

uniformly for every h in the interval

$$(3) \quad 0 \leq h \leq R_1 < R.$$

Now for (2) to have place it is obviously *sufficient* that the integrand in R_n , i. e., C_n , should uniformly vanish over the rectangle $[0, R_1; 0, 1]$. To prove that this condition is also *necessary* we proceed as follows. Since by hypothesis

$$f(x+h) = \sum_0^{\infty} f^{(n)}(x) \frac{h^n}{n!} \quad (0 \leq h < R),$$

differentiating p times with respect to h we get

$$f^{(p)}(x+h) = \sum_0^{\infty} f^{(n+p)}(x) \frac{h^n}{n!} \quad (0 \leq h < R);$$

i. e., if $f(x+h)$ is expansible by Taylor's series, so is $f^{(p)}(x+h)$. Thus taking $p = 1$ we have

$$\lim_{n=\infty} \int_0^1 h^n f^{(n+1)}(x+\theta h) \frac{(1-\theta)^{n-1}}{(n-1)!} d\theta = 0$$

uniformly for every $0 \leq h \leq R_1 < R$.

The fundamental lemma on which our proof rests is the following:

LEMMA. *If*

$$\lim_{n=\infty} \int_0^1 h^n f^{(n+1)}(x+\theta h) \frac{(1-\theta)^{n-1}}{(n-1)!} d\theta = 0$$

uniformly over $0 \leq h \leq R_1$, then

$$\lim_{n=\infty} \int_0^1 \left| h^n f^{(n+1)}(x+\theta h) \frac{(1-\theta)^{n-1}}{(n-1)!} \right| d\theta = 0$$

uniformly over $[0, R_1; 0, 1]$.

To prove this lemma it will evidently suffice to show that

$$\lim_{n=\infty} \int_0^1 \left| h^n f^{(n+1)}(x+\theta h) \frac{(1-\theta)^{n-1}}{(n-1)!} \right| d\theta = 0$$

uniformly over $0 \leq h \leq R_1$.

Replacing $f^{(n+1)}(x+\theta h)$ by $\sum_{p=0}^{\infty} f^{(n+p+1)}(x) \frac{(\theta h)^p}{p!}$ (its expansion by Taylor's series) we get

$$\begin{aligned} \int_0^1 \left| h^n f^{(n+1)}(x+\theta h) \frac{(1-\theta)^{n-1}}{(n-1)!} \right| d\theta & \\ & \leq \int_0^1 h^n \sum_{p=0}^{\infty} \left| f^{(n+p+1)}(x) \right| \frac{(\theta h)^p}{p!} \frac{(1-\theta)^{n-1}}{(n-1)!} d\theta \\ & \leq \sum_{p=0}^{\infty} \left| f^{(n+p+1)}(x) \right| \frac{h^{n+p}}{(n+p)!} .^* \end{aligned}$$

This last expression is the sum of the absolute values of the terms after the n^{th} in the Taylor's expansion of $f'(x+h)$, hence it vanishes uniformly over $0 \leq h \leq R_1$ when n becomes infinite, and the lemma is proved.

* We use here the well known formula

$$\int_0^1 \theta^p (1-\theta)^{n-1} d\theta = \frac{p!(n-1)!}{(n+p)!} .$$

To complete the proof, integrating by parts we have

$$\begin{aligned} h \int_0^1 h^n f^{(n+1)}(x + \theta h) \frac{(1 - \theta)^{n-1}}{(n-1)!} d\theta &= - h^n f^{(n)}(x + \theta h) \frac{(1 - \theta)^{n-1}}{(n-1)!} \\ &\quad - h \int_0^1 h^{n-1} f^{(n)}(x + \theta h) \frac{(1 - \theta)^{n-2}}{(n-2)!} d\theta, \end{aligned}$$

and since by the above lemma both of these integrals vanish uniformly over $[0, R_1; 0, 1]$, then so must

$$h^n f^{(n)}(x + \theta h) \frac{(1 - \theta)^{n-1}}{(n-1)!},$$

which proves the necessity of the criterion.

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